

# Large-Angle CMB Suppression and Polarisation Predictions

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## ABSTRACT

The anomalous lack of large angle temperature correlations has been a surprising feature of the CMB since first observed by *COBE-DMR* and subsequently confirmed and strengthened by *WMAP*. This anomaly may point to the need for modifications of the standard model of cosmology or may show that our Universe is a rare statistical fluctuation within that model. Further observations of the temperature auto-correlation function will not elucidate the issue; sufficiently high precision statistical observations already exist. Instead, alternative probes are required. In this work we explore the expectations for forthcoming polarisation observations. We define a prescription to test the hypothesis that the large-angle CMB temperature perturbations in our Universe represent a rare statistical fluctuation within the standard cosmological model. These tests are based on the temperature- $Q$  Stokes parameter correlation. Unfortunately these tests cannot be expected to be definitive. However, we do show that if this  $TQ$ -correlation is observed to be sufficiently large over an appropriately chosen angular range, then the hypothesis can be rejected at a high confidence level. We quantify these statements and optimise the statistics we have constructed to apply to the anticipated data. We find that we can construct a statistic that has a 25 per cent chance of excluding the hypothesis that we live in a rare realisation of  $\Lambda$ CDM at the 99.9 per cent confidence level.

**Key words:** cosmology: cosmic background radiation – cosmology: large-scale structure of Universe

## 1 INTRODUCTION

In the two decades since the Cosmic Background Explorer (*COBE*) first detected the primordial fluctuations in the cosmic microwave background (CMB) temperature (Wright et al. 1992), and perhaps even more so in the last decade over which the Wilkinson Microwave Anisotropy Probe (*WMAP*) has provided ever more accurate full-sky maps of those fluctuations (see Komatsu et al. 2011, for example), the CMB has become a keystone in the remarkable transition of cosmology from a qualitative to a precision science.

An important element of the role of the CMB in precision cosmology has been that the canonical theory of cosmology, inflationary Lambda Cold Dark Matter ( $\Lambda$ CDM), makes clear predictions for the statistical properties of the spherical harmonic coefficients of the temperature fluctuations,

$$a_{\ell m} \equiv \int Y_{\ell m}^*(\theta, \phi) T(\theta, \phi) d(\cos \theta) d\phi, \quad (1)$$

which are predicted to be statistically isotropic realisations of inde-

pendent Gaussian random variables of zero mean and with variance  $C_\ell$  depending only on  $\ell$ ,

$$C_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2. \quad (2)$$

In modern discussions of the CMB the two-point angular power spectrum, embodied in these  $C_\ell$ , plays a central role and is the source of the remarkable precision of the cosmological parameters (Komatsu et al. 2011).

Before the *COBE* era, it was the two-point angular correlation function of the fluctuations,

$$C(\theta) \equiv \overline{T(\hat{n}_1)T(\hat{n}_2)}|_{\hat{n}_1 \cdot \hat{n}_2 = \cos \theta}, \quad (3)$$

rather than the angular power spectrum that was of primary interest to astronomers. Statistically the two-point angular correlation function is an ensemble average but, in practice, this must be replaced by an average over pairs of points separated by an angle  $\theta$ , as denoted by the bar over the expression. In fact the *COBE* Differential Microwave Radiometer (*COBE-DMR*) did report  $C(\theta)$ , though only in their final, four-year paper (Bennett et al. 1996). When extracted from a full sky map both  $C(\theta)$  and the  $C_\ell$  contain the same information, albeit in different forms. The same is true between a function and its Fourier transform; signals are typically most easily

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seen in one or the other forms but not both. In the case of the CMB,  $C(\theta)$  and the  $C_\ell$  are related by a Legendre transform. The  $C_\ell$  most easily show the small angular scale, microphysics at last scattering; whereas  $C(\theta)$  most easily shows the large angular scale behaviour.

As observed by the *COBE-DMR*,  $C(\theta)$  had an unexpected property – it was consistent with zero for angular separations between approximately  $60^\circ$  and  $160^\circ$ . This was duly noted at the time but mainly remembered today as a low quadrupole. The *WMAP* team confirmed the *COBE-DMR* observation of a lack of large-angle correlation with significantly smaller error bars. In their initial, one-year release, Spergel et al. (2003) phrased the anomaly in terms of a statistic

$$S_{1/2} \equiv \int_{-1}^{1/2} [C(\theta)]^2 d(\cos \theta). \quad (4)$$

In the best fit  $\Lambda$ CDM model the expected value of this statistic is approximately  $50,000 (\mu\text{K})^2$ , whereas the observed value is approximately  $8500 (\mu\text{K})^2$  on the full sky, for example from the *WMAP* Independent Linear Combination (ILC) map,<sup>1</sup> with a  $p$ -value of approximately 0.05. Even more striking is that if one considers only the part of the sky outside a conservative galaxy cut, then  $S_{1/2} \simeq 1000 - 1150 (\mu\text{K})^2$  and is only  $\sim 1300 (\mu\text{K})^2$  in each of the  $V$  and  $W$  frequency bands, which are expected to be dominated by the CMB signal. The cut-sky  $S_{1/2}$  has a  $p$ -value less than about  $2.5 \times 10^{-4}$ , depending on the precise map (Copi et al. 2009).

We have argued that such absence of two-point angular correlation is unlikely to result solely from a small quadrupole, or even a small quadrupole and octopole, and that it instead requires a “conspiracy” among the first several multipoles (Copi et al. 2009). Such covariance among the  $C_\ell$  is likely contrary to the fundamental prediction of the canonical cosmological model that the  $a_{\ell m}$  underlying the  $C_\ell$  are Gaussian random independent variables with variances depending only on  $\ell$ . This is one of a number of large scale anomalies that suggest modifications of the standard model are required on large angular scales (see Bennett et al. 2011; Copi et al. 2010, and references therein for further details).

One possible explanation of the absence of large angle correlations among the CMB temperature fluctuations is that it is merely a statistical fluke. In this paper we explore the consequences of this hypothesis. In particular, since the fluctuations in the CMB temperature and in its polarisation arise largely from the same source – the gravitational potential – one might have hoped that a small temperature-temperature ( $TT$ ) correlation function on large angular scales would predict a similarly small cross-correlation between CMB temperature and CMB Stokes parameter  $Q$ , or in the polarisation-polarisation ( $QQ$ ) correlation.

Unfortunately, as we shall see, the connection between temperature and polarisation fluctuations is too weak for a general definitive test of the origin of the vanishing correlation function. However, we do find that if the  $S_{1/2}$  is small because of a statistical fluke within  $\Lambda$ CDM cosmology, then the cross correlation between temperature and polarisation is unlikely to be large on large angular scales. Therefore were we to infer a large value of this cross-correlation from future data, that would be evidence against a statistical fluke as an explanation of the vanishing  $TT$  correlation.

In this paper we provide a prescription to follow in order to test this hypothesis. In Section 2 we describe the construction of

an ensemble of realisations of  $\Lambda$ CDM that are constrained to resemble our observed universe in the properties of their  $TT$  angular power spectrum and full-sky and cut-sky two-point angular correlation functions. In Section 3 we construct statistics that, like  $S_{1/2}$  for the  $TT$  correlations, can be used to quantify the smallness of the correlations between temperature and polarisation fluctuations. Section 4 contains a discussion of the results of applying the temperature-polarisation cross-correlation statistics to the ensemble of constrained realisations and looks forward to what might be learned by applying them to future polarisation data. Given the constructed realisations the value of the statistics are fixed and the optimal application of them to polarisation data can be determined as discussed in this section. This optimisation is *independent* of polarisation observations. Finally, Section 5 contains the conclusions.

## 2 CONSTRAINED REALISATIONS

To study the signature of the lack of large angle correlations in the *WMAP* temperature data on upcoming polarisation measurements, such as from *Planck*, we require realisations of  $\Lambda$ CDM consistent with these large angle results. For this purpose we have generated 300,000 such realisations as follows.

(i) The  $C_\ell^{TT}$  at low- $\ell$  are well measured. Assuming Gaussian statistical errors we generate random  $C_\ell^{TT}$  from Gaussian distributions centred on the *WMAP*-reported values. This produces a power spectrum consistent with that reported by *WMAP*. It is important to stress that this is a power spectrum consistent with the observation of our particular realisation of the Universe as measured by *WMAP*, not a general realisation of the best fit  $\Lambda$ CDM model. For this reason cosmic variance is not relevant nor do we generate  $C_\ell$  from a  $\chi^2$  distribution. It is also true that on a partial sky the  $C_\ell$  are slightly correlated. To correct for this we actually use the Fisher matrix from the *WMAP* likelihood code *without* the contribution from cosmic variance in drawing these  $C_\ell^{TT}$ . In practice this is a small correction but has been included for completeness.

(ii) Given the  $C_\ell^{TT}$  from the previous part and assuming Gaussian random phases, a single sky realisation, the  $a_{\ell m}^T$ , is generated. The resulting sky realisation has a  $S_{1/2}$  consistent with the small value in the full sky *WMAP* ILC map.

(iii) To further be consistent with *WMAP* observations the  $S_{1/2}$  on the cut sky must also be small. For our realisations we require  $S_{1/2}^{\text{cut}} \leq 1292.6 (\mu\text{K})^4$ , the value from the *WMAP* seven-year, KQ75y7 masked ILC map. This  $S_{1/2}$  value is calculated for a realisation by first constructing a map at NSIDE=64 from the  $a_{\ell m}^T$  generated above. The pseudo- $C_\ell$  are extracted from this map based on the region outside the KQ75y7 mask using SPICE (Chon et al. 2004). Finally  $S_{1/2}$  is calculated using these  $C_\ell$  up to  $\ell_{\text{max}} = 100$ .

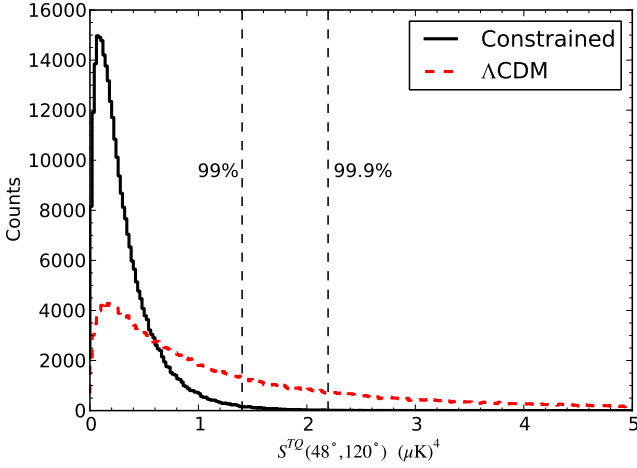
(iv) For temperature realisations that satisfy the cut sky constraint we also generate realisations of the  $a_{\ell m}^E$ . See Appendix A for a review of the process.

The construction of a set of constrained realisations is the basis for the prescription we are describing. It will next provide predictions for the expectations from the observations of the CMB polarisation.

## 3 STATISTICS

For the temperature auto-correlation the  $S_{1/2}$  statistic was defined *a posteriori* (Spergel et al. 2003) to be

<sup>1</sup> The ILC map and all data from the *WMAP* mission used in this work is freely available from <http://lambda.gsfc.nasa.gov/>.



**Figure 1.** Example histogram of the  $S^{TQ}$  statistic (6) for constrained (solid, black line) and  $\Lambda$ CDM (dashed, red line) realisations. We note that the constrained realisations are more sharply peaked at low  $S^{TQ}$  than  $\Lambda$ CDM which, though peaked at approximately the same value, has a long tail. The dashed, vertical lines represent the values of  $S^{TQ}$  for which 99 per cent and 99.9 per cent, respectively, of the constrained realisations have smaller values.

$$S_{1/2} \equiv \int_{-1}^{1/2} [C^{TT}(\theta)]^2 d(\cos \theta). \quad (5)$$

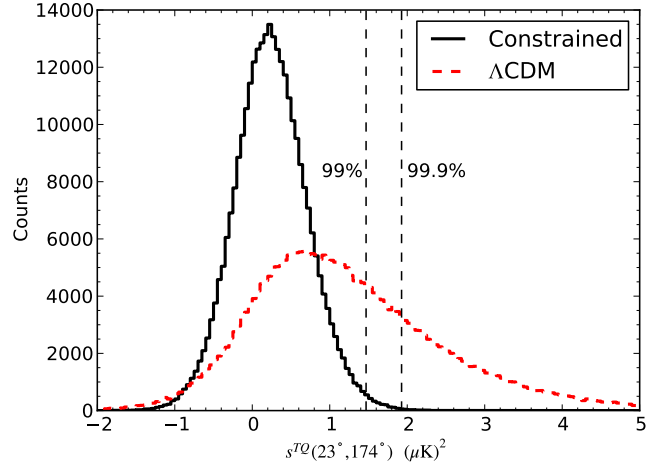
Inspired by this we define a comparable statistic for  $C^{TQ}(\theta)$ , the two-point angular correlation function between fluctuations in the temperature and the Stokes parameter  $Q$ .

Observable properties of photons can be characterised by the Stokes parameters. For the CMB the relevant quantities are the intensity, conventionally represented by the temperature,  $T$ , and the linear polarisation given by the  $Q$  and  $U$  parameters. For the CMB the circular polarisation, represented by the  $V$  Stokes parameter, is expected to be zero and not considered further. When working in real space the natural correlations to construct are amongst these observables,  $T$ ,  $Q$ , and  $U$ . These correlations are constructed such that they only depend on the angular separation between pairs of point on the sky and are thus rotationally invariant despite the fact that the definition of  $Q$  and  $U$  depend on the choice of coordinate axes. When working in harmonic space, it is natural to decompose the polarisation into “gradient” and “curl” modes alternatively called  $E$  and  $B$  modes, which are similarly rotationally invariant quantities. These latter names will be used throughout. Thus in real space we will work with the  $TQ$  two-point angular correlation function,  $C^{TQ}(\theta)$ , which may be written in terms of the two-point angular power spectrum coefficients,  $C_\ell^{TE}$ . Details of these representations can be found in standard references (Zaldarriaga & Seljak 1997; Kamionkowski et al. 1997, for example).

### 3.1 $S^{TQ}$ Statistic

In the case of polarisation *a priori* the optimal range over which to integrate the correlation function is unknown and will be explored below so we define the general statistic

$$S^{TQ}(\theta_1, \theta_2) \equiv \int_{\cos \theta_2}^{\cos \theta_1} [C^{TQ}(\theta)]^2 d(\cos \theta). \quad (6)$$



**Figure 2.** Example histogram of the  $s^{TQ}$  statistic (9) for constrained (solid, black line) and  $\Lambda$ CDM (dashed, red line) realisations. We note that the constrained realisations are more sharply peaked near zero than  $\Lambda$ CDM which, though also peaked near zero, has a long tail particularly to large, positive values. The dashed, vertical lines represent the values of  $s^{TQ}$  for which 99 per cent and 99.9 per cent, respectively, of the constrained realisations have smaller values.

As with  $S_{1/2}$  we may calculate this easily in terms of the power spectrum coefficients,  $C_\ell^{TE}$ . Using

$$C^{TQ}(\theta) = \sum_{\ell=2}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{\frac{(\ell-2)!}{(\ell+2)!}} C_\ell^{TE} P_\ell^2(\cos \theta) \quad (7)$$

we may show that

$$S^{TQ}(\theta_1, \theta_2) = \sum_{\ell, \ell'} C_\ell^{TE} I_{\ell, \ell'}^{TQ}(\theta_1, \theta_2) C_{\ell'}^{TE}, \quad (8)$$

where  $I_{\ell, \ell'}^{TQ}(\theta_1, \theta_2)$  are components of a known matrix calculated in Appendix B. A histogram of the  $S^{TQ}$  statistic for a particular choice of  $\theta_1$  and  $\theta_2$  is shown in Fig. 1 comparing the constrained realisations to  $\Lambda$ CDM.

### 3.2 $s^{TQ}$ Statistic

Motivated solely by its simplicity and ease of computation we also define a new statistic which is linear, rather than quadratic, in the  $TQ$  correlation function

$$s^{TQ}(\theta_1, \theta_2) \equiv \int_{\cos \theta_2}^{\cos \theta_1} C^{TQ}(\theta) d(\cos \theta). \quad (9)$$

As with  $S^{TQ}(\theta_1, \theta_2)$  we may calculate this easily in terms of the  $C_\ell^{TE}$ ,

$$s^{TQ}(\theta_1, \theta_2) = \sum_{\ell=2}^{\infty} C_\ell^{TE} i_\ell^{TQ}(\theta_1, \theta_2), \quad (10)$$

where  $i_\ell^{TQ}(\theta_1, \theta_2)$  are the components of a known vector calculated in Appendix B. A histogram of the  $s^{TQ}$  statistic for a particular choice of  $\theta_1$  and  $\theta_2$  is shown in Fig. 2 comparing the constrained realisations to  $\Lambda$ CDM.

#### 4 RESULTS

The  $S^{TQ}$  and  $s^{TQ}$  statistics defined above have been calculated for the constrained realisations discussed in Section 2 and for a comparable number of realisations of  $\Lambda$ CDM. In both cases these have been calculated from maps produced at NSIDE=64. Data from the temperature map outside the KQ75y7 mask and from the seven-year polarisation analysis mask, both provided by *WMAP*, has been used. As shown in Figs. 1 and 2 the constrained and  $\Lambda$ CDM realisations have most likely values for the  $S^{TQ}$  and  $s^{TQ}$  statistics at nearly the same value. However, as we also see  $\Lambda$ CDM predicts much broader distributions for the two statistics. In particular there is a significant probability of producing values larger than the constrained realisations. This provides a means of testing the hypothesis that our Universe is just a rare realisation of  $\Lambda$ CDM. This results in a simple but not definitive test.

Consider the case of the  $S^{TQ}$  statistic as represented in Fig. 1. For the constrained realisations 99 per cent of them have  $S^{TQ}(48^\circ, 120^\circ) \leq 1.403 (\mu\text{K})^4$  and 99.9 per cent have  $S^{TQ}(48^\circ, 120^\circ) \leq 2.195 (\mu\text{K})^4$ . Unconstrained  $\Lambda$ CDM (with the best fit values of cosmological parameters) randomly generates realisations with values larger than these 38.6 per cent and 25.6 per cent of the time, respectively. If observations of the polarisation show our Universe to have a  $S^{TQ}(48^\circ, 120^\circ)$  value larger than these values then we can reject the random  $\Lambda$ CDM realisation hypothesis at the appropriate confidence level. Alternatively, if the  $S^{TQ}(48^\circ, 120^\circ)$  value is smaller then no definitive statement can be made; the polarisation fluctuations would be consistent with the hypothesis that we live in a rare  $\Lambda$ CDM realisation but do nothing to advance that hypothesis. This is the main point of the paper.

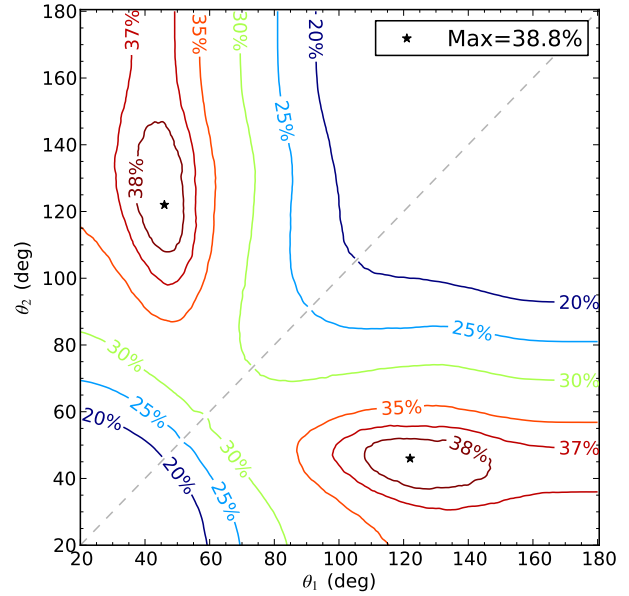
Similar statements may be made about the  $s^{TQ}$  statistic shown in Fig. 2. It provides similar information as  $S^{TQ}$ .

It remains to determine the optimal ranges over which to evaluate the statistics. We define optimal to mean the maximum discriminatory power between the distribution of the statistic in the constrained versus  $\Lambda$ CDM realisations. For a given per cent cutoff from the constrained realisations we wish to find the angle range  $[\theta_1, \theta_2]$  that has the maximum fraction of  $\Lambda$ CDM realisations above this value.

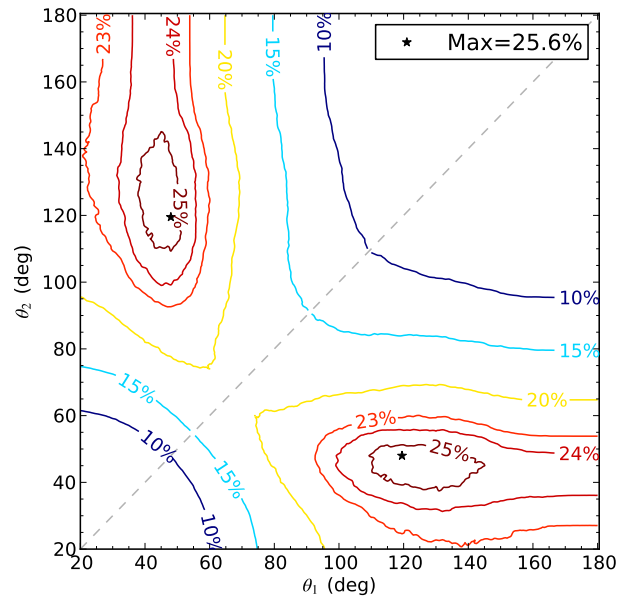
The results of such a study are shown in Figs. 3 and 4 for the  $S^{TQ}(\theta_1, \theta_2)$  statistic and in Figs. 5 and 6 for the  $s^{TQ}(\theta_1, \theta_2)$  statistic. The statistics are non-zero only up to the diagonal  $\theta_1 = \theta_2$ , shown as the dashed, grey line in the figures, but not along it. For this reason the contours are truncated at the diagonal. They have been made symmetric in  $\theta_1$  and  $\theta_2$  (by taking  $|S^{TQ}|$  and  $|s^{TQ}|$ ) so the results are shown as identical when reflected through the diagonal. The optimal ranges and fractions of  $\Lambda$ CDM realisations are listed in Table 1. We note that the optimal surfaces represented by the contours seen in the figures are relatively broad, at least in one direction. Due to this the values of  $\theta_1$  and  $\theta_2$  in a neighbourhood of those listed in Table 1 can be employed with equal efficacy.

#### 5 CONCLUSIONS

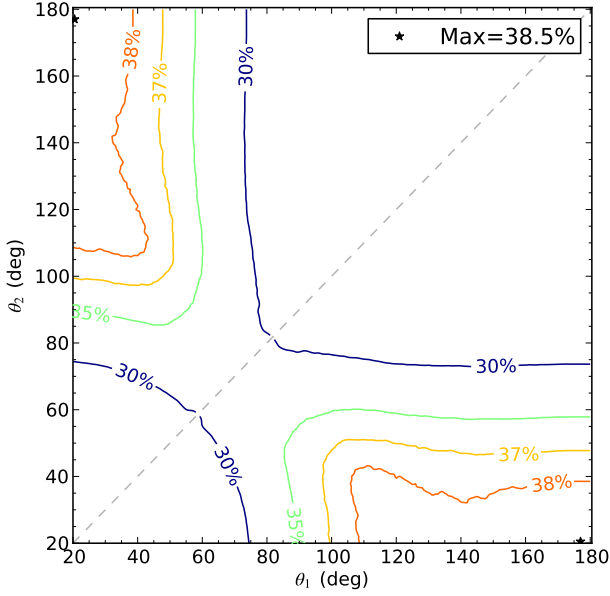
The absence of two-point angular correlations on large angular scales in the CMB temperature data is, by now, well established. It was first measured by *COBE-DMR*, but became more significant in the *WMAP* temperature maps. This absence is difficult to accommodate within the standard cosmological model, especially since it seems to imply co-variance among low- $\ell$  multipoles of the CMB. A simple explanation that has been proffered is that we just live in a



**Figure 3.** Contours for the fraction of  $\Lambda$ CDM realisations above the 99 per cent value of the constrained realisations from the  $S^{TQ}$  statistic (6). In the optimal case 38.8 per cent of the  $\Lambda$ CDM realisations have a larger value. In this figure the results around not defined along the diagonal (grey, dashed line) and have been made symmetric about it by taking the absolute value of the statistic.



**Figure 4.** The same as Fig. 3 now for the 99.9 per cent case. Here in the optimal case 25.6 per cent of the  $\Lambda$ CDM realisations have a larger value and the full histograms are shown in Fig. 1.



**Figure 5.** Similar to Fig. 3 but now for the  $s^{TQ}$  statistic (9). Here in the optimal case 38.5 per cent of the  $\Lambda$ CDM realisations have a larger value at the 99 per cent level.

**Table 1.** Optimal angle ranges for the  $S^{TQ}$  statistic (6) and  $s^{TQ}$  statistic (9). The optimal ranges are determined by finding the maximum fraction of  $\Lambda$ CDM realisations with the appropriate statistic above the 99 per cent or 99.9 per cent level of the constrained realisations. The histograms for the optimal 99.9 per cent ranges are shown in Figs. 1 and 2. The full contours are shown in Figs. 3–6.

Statistic	C.L. (per cent)	$\theta_1$ (degree)	$\theta_2$ (degree)	Fraction (per cent)
$S^{TQ}$	99	46	122	38.8
	99.9	48	120	25.6
$s^{TQ}$	99	20	177	38.5
	99.9	23	174	26.4

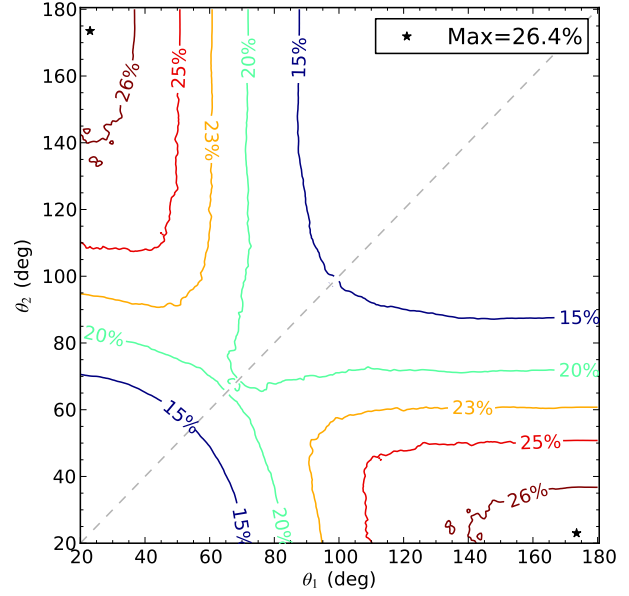
rare realisation of  $\Lambda$ CDM that happens to have a lack of large angle  $TT$  correlations. If so, one might hope that constraining  $\Lambda$ CDM realisations to have low  $TT$  correlations at large angles, would have observable consequences for other correlation functions, such as  $TQ$ . These would be the basis for an observational test of this statistical fluke hypothesis.

In this paper we have discussed a prescription to follow in order to test this hypothesis of our Universe being a statistical fluke. The prescription may be simply stated as follows.

(i) Construct realisations of our Universe consistent with the observed temperature fluctuations. This means construct sets of  $a_{\ell m}^T$  and  $a_{\ell m}^E$  consistent with the observed  $C_\ell^{TT}$  and full and cut sky  $S_{1/2}$  as discussed in Section 2.

(ii) Apply the  $S^{TQ}$  and  $s^{TQ}$  statistics as defined in Section 3 to these constrained realisations.

(iii) Also apply the  $S^{TQ}$  and  $s^{TQ}$  statistics to a comparable number of best fit  $\Lambda$ CDM realisations and use these to find the optimal ranges  $[\theta_1, \theta_2]$  for each statistic. Optimal here means that



**Figure 6.** The same as Fig. 5 now for the 99.9 per cent case. Here in the optimal case 26.4 per cent of the  $\Lambda$ CDM realisations have a larger value and the full histograms are shown in Fig. 2.

the maximum fraction of  $\Lambda$ CDM realisations fall above the value at some confidence level in the constrained realisation, for example the 99 per cent or 99.9 per cent level.

(iv) Given the optimal ranges from the previous step now apply these particular cases to the observed polarisation signal. If the observations produce values for these statistics larger than that expected from the constrained realisations, then the statistical fluke hypothesis can be rejected at the appropriate confidence level. Alternatively if the values are larger, then the hypothesis remains consistent but unproven.

We further note that the optimisation in this prescription is independent of the polarisation observations, or, in fact, whether the polarisation has been observed or not.

Our work is, in spirit, related to Dvorkin et al. (2008). Whereas they consider observables in the polarisation signal for “models” of three dimensional primordial power modulation that might explain the breaking of statistical isotropy in the temperature field, we predict the polarisation statistics starting directly from realisations of  $\Lambda$ CDM models that are constrained to show the suppressed  $TT$  correlation at large angular scales.

In the work reported here we have generated realisations and performed the optimisation based on the *WMAP* seven-year data release. The prescription could be applied to the *WMAP* nine-year data release and the results are not expected to differ significantly. We have also said nothing about applying the statistics to the *WMAP* reported polarisation observations. Unfortunately the signal-to-noise in the polarisation observations is not yet sufficient to make meaningful statements. To see this, using the *WMAP* nine-year reported  $C_\ell^{TE}$  (Hinshaw et al. 2012) we find for the optimal ranges

$$S_{WMAP}^{TQ}(48^\circ, 120^\circ) = (1.0 \pm 0.8) (\mu K)^4 \quad (11)$$

and

$$s_{WMAP}^{TQ}(23^\circ, 174^\circ) = (0.8 \pm 0.8) (\mu\text{K})^2. \quad (12)$$

Here the error bars are crude estimates assuming the reported  $C_\ell^{TE}$  are statistically independent and the noise is Gaussian. These assumptions are not justified and a more careful assessment could be performed using the Fisher matrix. However, given the large estimated errors such an assessment is not warranted.

We have shown that the prescription described in this work is not a definitive test of the statistical fluke hypothesis for our Universe. Nevertheless, by carefully optimising the statistical measure of large angle  $TQ$  correlations, we were able to demonstrate that once good  $TQ$  correlation data is available there is a reasonable probability (over 25 per cent) to reject the statistical fluke hypothesis at the 99.9 per cent confidence level. *WMAP* data is not up to this task, however *Planck* data should be.

## ACKNOWLEDGMENTS

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## APPENDIX A: POLARIZATION GAUSSIAN RANDOM REALISATIONS

The generation of correlated Gaussian random variables is a well known topic. For use in the CMB this is implemented in HEALPIX (Górski et al. 2005), for example. Here we review the details relevant for the generation of our constrained realisations.

In  $\Lambda$ CDM the temperature and  $E$ -type polarization are correlated as encoded in the power spectrum coefficients  $C_\ell^{TT}$ ,  $C_\ell^{TE}$ ,

and  $C_\ell^{EE}$ . Working in the real spherical harmonic basis we may generate the spherical harmonic coefficients as

$$a_j^T = \sqrt{C_\ell^{TT}} \zeta_1, \quad (A1)$$

$$a_j^E = \frac{C_\ell^{TE}}{\sqrt{C_\ell^{TT}}} \zeta_1 + \sqrt{C_\ell^{EE} - \frac{(C_\ell^{TE})^2}{C_\ell^{TT}}} \zeta_2, \quad (A2)$$

where  $\zeta_1$  and  $\zeta_2$  are Gaussian random variables drawn from a distribution with zero mean and unit variance.

For our purposes we are generating constrained realisations of  $\Lambda$ CDM so the  $C_\ell$  in the above expressions are determined from the best fit  $\Lambda$ CDM model consistent with *WMAP*. Since we have generated constrained realisations of the  $a_{\ell m}^T$  we have, effectively, chosen  $\zeta_1$  in (A1). We are left with generating the  $a_{\ell m}^E$ . The real and imaginary components of it are thus generated from (A2) using our known  $\zeta_1$  as

$$a_j^E = \frac{C_\ell^{TE}}{C_\ell^{TT}} a_j^T + \sqrt{C_\ell^{EE} - \frac{(C_\ell^{TE})^2}{C_\ell^{TT}}} \zeta_2, \quad (A3)$$

where  $\zeta_2$  is still to be chosen at random.

## APPENDIX B: DERIVATION OF STATISTICS FORMULAS

### B1 $S^{TQ}(\theta_1, \theta_2)$

We wish to evaluate  $S^{TQ}(\theta_1, \theta_2)$  as discussed in the text (6). Consider the simpler case

$$\begin{aligned} S^{TQ}(x) &\equiv \int_{-1}^x [C^{TQ}(\theta)]^2 d(\cos \theta) \\ &= \sum_{\ell, \ell'} \frac{(2\ell+1)(2\ell'+1)}{(4\pi)^2} \sqrt{\frac{(\ell-2)!(\ell'-2)!}{(\ell+2)!(\ell'+2)!}} C_\ell^{TE} C_{\ell'}^{TE} \\ &\quad \times \int_{-1}^x P_\ell^2(\cos \theta) P_{\ell'}^2(\cos \theta) d(\cos \theta). \end{aligned} \quad (B1)$$

To evaluate this expression we need to perform the integral of two associated Legendre functions,  $P_\ell^m$ , of order  $m = 2$ ,

$$\tilde{I}_{\ell, \ell'}^{TQ}(x) \equiv \int_{-1}^x P_\ell^2(x) P_{\ell'}^2(x) dx. \quad (B2)$$

Note that this integral is only defined for  $\ell, \ell' \geq 2$ .

For  $\ell \neq \ell'$  we may proceed by directly integrating the associated Legendre differential equation to find

$$\begin{aligned} \tilde{I}_{\ell, \ell'}^{TQ}(x) &= \frac{1-x^2}{\ell(\ell+1) - \ell'(\ell'+1)} \\ &\quad \times \left[ P_\ell^2(x) \frac{dP_{\ell'}^2(x)}{dx} - P_{\ell'}^2(x) \frac{dP_\ell^2(x)}{dx} \right]. \end{aligned} \quad (B3)$$

For  $\ell = \ell'$  more care is required. Starting from the Rodriguez formula

$$P_\ell^2(x) = (1-x^2) \frac{d^2 P_\ell(x)}{dx^2} \quad (B4)$$

and the recursion relation

$$x \frac{dP_\ell(x)}{dx} = \frac{dP_{\ell-1}(x)}{dx} + \ell P_\ell(x) \quad (B5)$$

we can show that

$$\tilde{I}_{\ell, \ell}^{TQ}(x) = 4J_{\ell-1}^{(2)}(x) - 4\ell(\ell-1)J_\ell^{(1)}(x) + \ell^2(\ell-1)^2 \tilde{I}_{\ell, \ell}(x). \quad (B6)$$

Here  $\tilde{I}_{\ell,\ell}(x)$  is the equivalent integral over Legendre polynomials encountered in the definition of  $S_{1/2}$ ; see Appendix A of Copi et al. (2009) for details. The remaining quantities,  $J_\ell^{(1)}(x)$  and  $J_\ell^{(2)}(x)$ , are calculated through integration by parts and use of the recursion relation to find

$$J_\ell^{(1)}(x) = P_{\ell-1}(x)P_\ell(x) + \frac{1}{2} \left\{ 1 - x [P_{\ell-1}(x)]^2 - (2\ell - 1)\tilde{I}_{\ell-1,\ell-1}(x) \right\}, \quad (\text{B7})$$

and

$$J_\ell^{(2)}(x) = J_{\ell-1}^{(2)}(x) + \ell [P_{\ell-1}(x)P_\ell(x) + 1]. \quad (\text{B8})$$

Notice that  $J_\ell^{(2)}(x)$  is defined recursively. We can directly show that  $J_0^{(2)}(x) = 0$ .

With these expressions for the integrals we may write

$$S^{TQ}(\theta_1, \theta_2) = \sum_{\ell, \ell'} C_\ell^{TE} I_{\ell, \ell'}^{TQ}(\theta_1, \theta_2) C_{\ell'}^{TE}, \quad (\text{B9})$$

where

$$I_{\ell, \ell'}^{TQ}(\theta_1, \theta_2) = \frac{(2\ell + 1)(2\ell' + 1)}{(4\pi)^2} \sqrt{\frac{(\ell - 2)!(\ell' - 2)!}{(\ell + 2)!(\ell' + 2)!}} \times [\tilde{I}_{\ell, \ell'}^{TQ}(\cos \theta_1) - \tilde{I}_{\ell, \ell'}^{TQ}(\cos \theta_2)]. \quad (\text{B10})$$

This matrix may be precomputed for rapid evaluation of  $S^{TQ}(\theta_1, \theta_2)$ .

## B2 $s^{TQ}(\theta_1, \theta_2)$

We wish to evaluate  $s^{TQ}(\theta_1, \theta_2)$  as discussed in the text (9). We proceed as above and consider the simpler case

$$\begin{aligned} s^{TQ}(x) &\equiv \int_{-1}^x C^{TQ}(\theta) d(\cos \theta) \\ &= \sum_{\ell=2}^{\infty} \frac{(2\ell + 1)}{4\pi} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} C_\ell^{TE} \\ &\quad \times \int_{-1}^x P_\ell^2(\cos \theta) d(\cos \theta). \end{aligned} \quad (\text{B11})$$

To evaluate this expression we need to perform the integral

$$\tilde{i}_\ell^{TQ}(x) \equiv \int_{-1}^x P_\ell^2(x) dx. \quad (\text{B12})$$

This integral is straight forward to evaluate starting from the Rodriguez formula (B4) and integrating by parts to find

$$\begin{aligned} \tilde{i}_\ell^{TQ}(x) &= \ell P_{\ell-1}(x) - (\ell - 2)xP_\ell(x) + 2(-1)^\ell \\ &\quad - \frac{2}{2\ell + 1} [P_{\ell+1}(x) - P_{\ell-1}(x)]. \end{aligned} \quad (\text{B13})$$

With this we may write

$$s^{TQ}(\theta_1, \theta_2) = \sum_{\ell=2}^{\infty} C_\ell^{TE} \tilde{i}_\ell^{TQ}(\theta_1, \theta_2) \quad (\text{B14})$$

where

$$\tilde{i}_\ell^{TQ}(\theta_1, \theta_2) = \frac{2\ell + 1}{4\pi} \sqrt{\frac{(\ell - 2)!}{(\ell + 2)!}} [\tilde{i}_\ell^{TQ}(\cos \theta_1) - \tilde{i}_\ell^{TQ}(\cos \theta_2)]. \quad (\text{B15})$$

This vector may be precomputed for rapid evaluation of  $s^{TQ}(\theta_1, \theta_2)$ .

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